Curvatures of metrics induced by the isotropic almost complex structures

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Abstract

Isotropic almost complex structures induce a class of Riemannian metrics on tangent bundle of a Riemannian manifold. In this paper the curvature tensors of these metrics will be calculated.

Keywords: Isotropic almost complex structure, tangent bundle, curvature tensor.

1 Introduction

Let (M, g) be a Riemannian manifold and $\pi : TM \to M$ be its tangent bundle. Moreover, denote by X^h, X^v the horizontal and vertical lifts of the vector field X on M. In [2], Aguilar defined a class of almost complex structures $J_{\delta,\sigma}$ on TM, namely isotropic almost complex structures with definition

$$J_{\delta,\sigma}(X^h) = \alpha X^v + \sigma X^h, \qquad J_{\delta,\sigma}(X^v) = -\sigma X^v - \delta X^h, \tag{1}$$

for functions $\alpha, \delta, \sigma : TM \to \mathbb{R}$ which are in relation $\alpha\delta - \sigma^2 = 1$.

He studied the integrability of these structures and proved that there exists an integrable isotropic almost complex structure on the tangent bundle of a

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Riemannian manifold (M, g) if and only if the sectional curvature of (M, g) is constant.

Also, using the Liouville one-form, he defined a new class of Riemannian metrics $g_{\delta,\sigma}$ which are generalization of Sasaki metric. These metrics for an almost complex structure $J_{\delta,\sigma}$ is defined by

$$g_{\delta,\sigma}(X^h, Y^h) = \alpha g(X, Y) o\pi, \tag{2}$$
$$g_{\delta,\sigma}(X^h, Y^v) = -\sigma g(X, Y) o\pi, \tag{3}$$
$$g_{\delta,\sigma}(X^v, Y^v) = \delta g(X, Y) o\pi. \tag{4}$$

for vector fields X, Y on M.

2 Curvatures of $g_{\delta,0}$

The following theorem states the formulas of the Levi-Civita connection of $g_{\delta,\sigma}$.

Theorem 1 *Let $g_{\delta,\sigma}$ be a Riemannian metric on TM as before. Then the Levi-Civita connection $\bar{\nabla}$ of $g_{\delta,\sigma}$ at $(p, u) \in TM$ is given by*

$$\begin{aligned}\bar{\nabla}_{X^h} Y^h = {} & (\nabla_X Y)^h - \frac{\sigma}{\alpha}(R(u, X)Y)^h + \frac{1}{2\alpha} X^h(\alpha) Y^h + \frac{1}{2\alpha} Y^h(\alpha) X^h \\ & - \frac{\sigma}{\delta}(\nabla_X Y)^v - \frac{1}{2}(R(X, Y)u)^v - \frac{1}{2\delta} X^h(\sigma) Y^v \\ & - \frac{1}{2\delta} Y^h(\sigma) X^v - \frac{1}{2} g(X, Y) \bar{\nabla}\alpha,\end{aligned} \tag{5}$$

$$\begin{aligned}\bar{\nabla}_{X^h} Y^v = {} & -\frac{\sigma}{\alpha}(\nabla_X Y)^h + \frac{\delta}{2\alpha}(R(u, Y)X)^h - \frac{1}{2\alpha} X^h(\sigma) Y^h \\ & + \frac{1}{2\alpha} Y^v(\alpha) X^h + (\nabla_X Y)^v + \frac{1}{2\delta} X^h(\delta) Y^v - \frac{1}{2\delta} Y^v(\sigma) X^v \\ & + \frac{1}{2} g(X, Y) \bar{\nabla}\sigma,\end{aligned} \tag{6}$$

$$\begin{aligned}\bar{\nabla}_{X^v} Y^h = {} & \frac{\delta}{2\alpha}(R(u, X)Y)^h + \frac{1}{2\alpha} X^v(\alpha) Y^h - \frac{1}{2\alpha} Y^h(\sigma) X^h \\ & - \frac{1}{2\delta} X^v(\sigma) Y^v + \frac{1}{2\delta} Y^h(\delta) X^v + \frac{1}{2} g(X, Y) \bar{\nabla}\sigma,\end{aligned} \tag{7}$$

$$\begin{aligned}\bar{\nabla}_{X^v} Y^v = {} & -\frac{1}{2\alpha} X^v(\sigma) Y^h - \frac{1}{2\alpha} Y^v(\sigma) X^h + \frac{1}{2\delta} X^v(\delta) Y^v \\ & + \frac{1}{2\delta} Y^v(\delta) X^v - \frac{1}{2} g(X, Y) \bar{\nabla}\delta.\end{aligned} \tag{8}$$

Proof. We just prove (5), the remaining ones are similar. Using Koszul formula, we have

$$\begin{aligned}2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^h, Z^h) &= X^h g_{\delta,\sigma}(Y^h, Z^h) + Y^h g_{\delta,\sigma}(X^h, Z^h) - Z^h g_{\delta,\sigma}(X^h, Y^h)\\ &+ g_{\delta,\sigma}([X^h, Y^h], Z^h) + g_{\delta,\sigma}([Z^h, X^h], Y^h)\\ &- g_{\delta,\sigma}([Y^h, Z^h], X^h).\end{aligned}$$

Using relations (14), (19) and (28) gives us

$$\begin{aligned}2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^h, Z^h) &= X^h(\alpha)g(Y, Z) + \alpha X g(Y, Z) + Y^h(\alpha)g(X, Z)\\ &+ \alpha Y g(X, Z) - Z^h(\alpha)g(X, Y) - \alpha Z g(X, Y)\\ &+ \alpha g([X, Y], Z) + \sigma g(R(X, Y)u, Z) + \alpha g([Z, X]Y)\\ &+ \sigma g(R(Z, X)u, Y) - \alpha g([Y, Z], X)\\ &- \sigma g(R(Y, Z)u, X).\end{aligned}$$

Using the properties of the Levi-Civita connection of g, we can get

$$\begin{aligned}2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^h, Z^h) &= g(X^h(\alpha)Y, Z) + g(Y^h(\alpha)X, Z) - Z^h(\alpha)g(X, Y)\\ &+ 2\alpha g(\nabla_X Y, Z) + \sigma g(R(X, Y)u, Z)\\ &+ \sigma g(R(Z, X)u, Y) - \sigma g(R(Y, Z)u, X).\end{aligned}$$

Taking into account (19) and the Bianchis first identity, we have

$$\begin{aligned}2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^h, Z^h) &= g_{\delta,\sigma}(\frac{1}{\alpha}X^h(\alpha)Y^h + \frac{1}{\alpha}Y^h(\alpha)X^h - g(X, Y)\bar{\nabla}\alpha\\ &+ 2(\nabla_X Y)^h - \frac{2\sigma}{\alpha}(R(u, X)Y)^h, Z^h),\end{aligned}$$

so the horizontal component of $\bar{\nabla}_{X^h}Y^h$ is

$$\begin{aligned}h(\bar{\nabla}_{X^h}Y^h) &= \frac{1}{2\alpha}X^h(\alpha)Y^h + \frac{1}{2\alpha}Y^h(\alpha)X^h - \frac{1}{2}g(X, Y)h(\bar{\nabla}\alpha) + (\nabla_X Y)^h\\ &- \frac{\sigma}{\alpha}(R(u, X)Y)^h,\end{aligned}$$

where $\bar{\nabla}\alpha = h(\bar{\nabla}\alpha) + v(\bar{\nabla}\alpha)$ is the splitting of the gradient vector field of α with respect to $g_{\delta,\sigma}$ to horizontal and vertical components, respectively. Similarly the vertical component of $\bar{\nabla}_{X^h}Y^h$ is

$$\begin{aligned}v(\bar{\nabla}_{X^h}Y^h) &= -\frac{1}{2\delta}X^h(\sigma)Y^v - \frac{1}{2\delta}Y^h(\sigma)X^v - \frac{1}{2}g(X, Y)v(\bar{\nabla}\alpha)\\ &- \frac{\sigma}{\delta}(\nabla_X Y)^v - \frac{1}{2}(R(X, Y)u)^v.\end{aligned}$$

Using the equation $(\bar{\nabla}_{X^h}Y^h) = h(\bar{\nabla}_{X^h}Y^h) + v(\bar{\nabla}_{X^h}Y^h)$, the proof will be completed. ■

Hereafter, we put $\sigma = 0$ and represent the metric $g_{\delta,0}$ by $\bar{g}$ and its Levi-Civita connection by $\bar{\nabla}$.

Definition 2 *Let (M,g) be a Riemannian manifold and ∇ be the Levi-Civita connection of g. Moreover, let $C^\infty M$ be the set of all smooth functions on M. The differential operator $\Delta_g : C^\infty M \longrightarrow C^\infty M$ given by*

$$\Delta_g(f) = \Sigma_{i=1}^n \{\nabla_{E_i}\nabla_{E_i}(f) - \nabla_{\nabla_{E_i}E_i}(f)\},$$

is called rough Laplacian on functions, where $\{E_1,\dots,E_n\}$ is a locally orthonormal frame on M and $f \in C^\infty M$.

Note that in some books this operator is defined with a minus sign. Let $\{E_1,\dots,E_n\}$ be a locally orthonormal frame on (M,g) around $p \in M$ such that $\nabla_{E_i}E_j = 0$ at p. Then, it is obvious that $\{\frac{E_1^h}{\sqrt{\alpha}},\dots,\frac{E_1^h}{\sqrt{\alpha}},\sqrt{\alpha}E_1^v,\dots,\sqrt{\alpha}E_n^v\}$ is a locally orthonormal frame on $(\mathrm{T}M,\bar{g})$. The Laplacian of α at p is calculated as follow,

Lemma 3 *Using the above notations $\Delta_{\bar{g}}\alpha$ is given by*

$$\begin{aligned}\Delta_{\bar{g}}\alpha(p) = \sum_{i=1}^n \{&\frac{1}{\alpha}E_i^h(E_i^h(\alpha)) + \alpha E_i^v(E_i^v(\alpha)) \\ &- \frac{1}{\alpha^2}E_i^h(\alpha)E_i^h(\alpha) + E_i^v(\alpha)E_i^v(\alpha)\}(p).\end{aligned}$$

Proof. Using the definition 2 gives us,

$$\begin{aligned}\Delta_{\bar{g}}\alpha(p) &= \Sigma_{i=1}^n\{\frac{1}{\sqrt{\alpha}}E_i^h(\frac{1}{\sqrt{\alpha}}E_i^h(\alpha)) + \sqrt{\alpha}E_i^v(\sqrt{\alpha}E_i^v(\alpha)) \\ &- (\bar{\nabla}_{\frac{1}{\sqrt{\alpha}}E_i^h}\frac{1}{\sqrt{\alpha}}E_i^h)(\alpha) - (\bar{\nabla}_{\sqrt{\alpha}E_i^v}\sqrt{\alpha}E_i^v)(\alpha)\}(p) \\ &= \Sigma_{i=1}^n\{\frac{1}{\alpha}E_i^h(E_i^h(\alpha)) + \alpha E_i^v(E_i^v(\alpha)) \\ &- \frac{1}{\alpha}(\bar{\nabla}_{E_i^h}E_i^h)(\alpha) - \alpha(\bar{\nabla}_{E_i^v}E_i^v)(\alpha)\}(p).\end{aligned} \tag{9}$$

By putting the equations Levi-Civita in the equation (9), we get the result. ■

Let $u = u^i\frac{\partial}{\partial x^i} \in \mathrm{T}M$ be any vector in $\mathrm{T}M$. Then the following relations hold for every vector fields $X = X^i\frac{\partial}{\partial x^i}$, $Y = Y^i\frac{\partial}{\partial x^i}$ and $Z = Z^i\frac{\partial}{\partial x^i}$ on M [1]:

$$\begin{aligned}&X^h(u^i) = -\Gamma_{ts}^i X^t u^s, \quad \textit{and} \quad X^v(u^i) = X^i, \\ &X^h g(Y,u) = g(\nabla_X Y, u), \quad \textit{and} \quad X^v g(Y,u) = g(X,Y), \\ &X^h g(Y,Z) = Xg(Y,Z), \quad \textit{and} \quad X^v g(Y,Z) = 0,\end{aligned}$$

and the following derivatives are given in a computable form by using the two

first of above equations,

$$\bar{\nabla}_{X^h_u}(R(u,Y)Z)^h = u^i\bar{\nabla}_{X^h_u}(R(\frac{\partial}{\partial x^i},Y)Z)^h - u^i(R(\nabla_X\frac{\partial}{\partial x^i},Y)Z)^h_u, \tag{10}$$

$$\bar{\nabla}_{X^v_u}(R(u,Y)Z)^h = u^i\bar{\nabla}_{X^v_u}(R(\frac{\partial}{\partial x^i},Y)Z)^h + (R(X,Y)Z)^h_u, \tag{11}$$

$$\bar{\nabla}_{X^h_u}(R(u,Y)Z)^v = u^i\bar{\nabla}_{X^h_u}(R(\frac{\partial}{\partial x^i},Y)Z)^v - u^i(R(\nabla_X\frac{\partial}{\partial x^i},Y)Z)^v_u, \tag{12}$$

$$\bar{\nabla}_{X^v_u}(R(u,Y)Z)^v = u^i\bar{\nabla}_{X^v_u}(R(\frac{\partial}{\partial x^i},Y)Z)^v + (R(X,Y)Z)^v_u. \tag{13}$$

The curvatures of $\bar{g}$ can be calculated using the following formulas

$$\bar{R}(A,B)C = \bar{\nabla}_A\bar{\nabla}_B C - \bar{\nabla}_B\bar{\nabla}_A C - \bar{\nabla}_{[A,B]}C,$$
$$\bar{Q}(A) = \frac{1}{\alpha}\Sigma_{i=1}^n\bar{R}(A,E_i^h)E_i^h + \alpha\bar{R}(A,E_i^v)E_i^v,$$

for the Riemannian curvature tensor and the Ricci operator, respectively, where $\{\frac{1}{\sqrt{\alpha}}E_1^h,\ldots,\frac{1}{\sqrt{\alpha}}E_n^h,\sqrt{\alpha}E_1^v,\ldots,\sqrt{\alpha}E_n^v\}$ represents a locally orthonormal frame for $(\mathrm{T}M,\bar{g})$ and A, B and C are vector fields on $\mathrm{T}M$.

Theorem 4 *Let (M,g) be a Riemannian manifold and $\bar{g}$ be a Riemannian metric induced by $J_{\delta,0}$ on $\mathrm{T}M$. Denote by ∇ and R the Levi-Civita connection and the Riemannian curvature tensor of (M,g), respectively. Then the Riemannian curvature tensors of $(\mathrm{T}M,\bar{g})$ are completely determined by*

$$\begin{aligned}
\bar{R}(X^h,Y^h)Z^h = {} & (R(X,Y)Z)^h - \frac{1}{4\alpha^2}(R(u,R(Y,Z)u)X)^h \\
& + \frac{1}{4\alpha^2}(R(u,R(X,Z)u)Y)^h + \frac{1}{2\alpha^2}(R(u,R(X,Y)u)Z)^h \\
& + \{\frac{1}{2\alpha}(\nabla_Y Z)^h(\alpha) + \frac{3}{4\alpha^2}Y^h(\alpha)Z^h(\alpha) - \frac{1}{2\alpha}Y^h(Z^h(\alpha)) \\
& - \frac{1}{4\alpha}(R(Y,Z)u)^v(\alpha)\}X^h + \{-\frac{1}{2\alpha}(\nabla_X Z)^h(\alpha) \\
& - \frac{3}{4\alpha^2}X^h(\alpha)Z^h(\alpha) + \frac{1}{2\alpha}X^h(Z^h(\alpha)) \\
& + \frac{1}{4\alpha}(R(X,Z)u)^v(\alpha)\}Y^h \\
& + \frac{1}{2}((\nabla_Z R)(X,Y)u)^v - \frac{1}{2\alpha}Y^h(\alpha)(R(X,Z)u)^v \\
& + \frac{1}{2\alpha}X^h(\alpha)(R(Y,Z)u)^v - \frac{1}{\alpha}Z^h(\alpha)(R(X,Y)u)^v \\
& + \{\frac{1}{4\alpha}X^h(\alpha)g(Y,Z) - \frac{1}{4\alpha}Y^h(\alpha)g(X,Z)\}\bar{\nabla}\alpha \\
& + \frac{1}{2}g(X,Z)\bar{\nabla}_{Y^h}\bar{\nabla}\alpha - \frac{1}{2}g(Y,Z)\bar{\nabla}_{X^h}\bar{\nabla}\alpha,
\end{aligned} \tag{14}$$

$$
\begin{aligned}
\bar{R}(X^h,Y^h)Z^v &= \frac{1}{2\alpha^2}((\nabla_X R)(u,Z)Y)^h - \frac{1}{2\alpha^2}((\nabla_Y R)(u,Z)X)^h \\
&+ \frac{1}{2\alpha^3}Y^h(\alpha)(R(u,Z)X)^h - \frac{1}{2\alpha^3}X^h(\alpha)(R(u,Z)Y)^h \\
&+ \{\frac{1}{4\alpha^3}(R(u,Z)Y)^h(\alpha) + \frac{1}{2\alpha}(\nabla_Y Z)^v(\alpha) \\
&+ \frac{1}{4\alpha^2}Y^h(\alpha)Z^v(\alpha) - \frac{1}{2\alpha}Y^h(Z^v(\alpha))\}X^h \\
&+ \{-\frac{1}{4\alpha^3}(R(u,Z)X)^h(\alpha) - \frac{1}{2\alpha}(\nabla_X Z)^v(\alpha) \\
&- \frac{1}{4\alpha^2}X^h(\alpha)Z^v(\alpha) + \frac{1}{2\alpha}X^h(Z^v(\alpha))\}Y^h \\
&+ (R(X,Y)Z)^v + \frac{1}{4\alpha^2}(R(u,Z).R)(X,Y)u)^v \\
&- \frac{1}{4\alpha^2}(R(u,Z)R(X,Y)u)^v + \frac{1}{4\alpha^2}(R(X,Y)R(u,Z)u)^v \\
&+ \frac{1}{\alpha}Z^v(\alpha)(R(Y,X)u)^v - \frac{1}{2\alpha}Z^v(\alpha)[X,Y]^v \\
&+ \frac{1}{\alpha^2}R(X,Y,u,Z)\bar{\nabla}\alpha,
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
\bar{R}(X^h,Y^v)Z^h &= \frac{1}{2\alpha^2}((\nabla_X R)(u,Y)Z)^h - \frac{1}{\alpha^3}X^h(\alpha)(R(u,Y)Z)^h \\
&- \frac{1}{2\alpha^3}Z^h(\alpha)(R(u,Y)X)^h + \{\frac{1}{4\alpha^3}(R(u,Y)Z)^h(\alpha) \\
&+ \frac{1}{4\alpha^2}Z^h(\alpha)Y^v(\alpha) - \frac{1}{2\alpha}Y^v(Z^h(\alpha))\}X^h \\
&+ \frac{1}{2}(R(X,Z)Y)^v - \frac{1}{4\alpha^2}(R(X,R(u,Y)Z)u)^v \\
&- \frac{1}{2\alpha}Y^v(\alpha)(R(X,Z)u)^v + \{-\frac{1}{2\alpha}X^h(Z^h(\alpha)) \\
&+ \frac{1}{2\alpha}(\nabla_X Z)^h(\alpha) + \frac{5}{4\alpha^2}Z^h(\alpha)X^h(\alpha) \\
&- \frac{1}{4\alpha}(R(X,Z)u)^v(\alpha)\}Y^v \\
&+ \{-\frac{1}{2\alpha^2}R(u,Y,Z,X) - \frac{1}{4\alpha}Y^v(\alpha)g(X,Z)\}\bar{\nabla}\alpha \\
&+ \frac{1}{2}g(X,Z)\bar{\nabla}_{Y^v}\bar{\nabla}\alpha,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\bar{R}(X^v, Y^h)Z^v = &\frac{1}{2\alpha^2}(R(X,Z)Y)^h + \frac{1}{4\alpha^4}(R(u,X)R(u,Z)Y)^h \\
&- \frac{1}{2\alpha^3}X^v(\alpha)(R(u,Z)Y)^h + \frac{1}{2\alpha^3}Z^v(\alpha)(R(u,X)Y)^h \\
&+ \{\frac{1}{4\alpha^2}X^v(\alpha)Z^v(\alpha) + \frac{1}{2\alpha}X^v(Z^v(\alpha))\}Y^h \\
&\{-\frac{1}{4\alpha^3}(R(u,Z)Y)^h(\alpha) - \frac{1}{2\alpha}(\nabla_Y Z)^v(\alpha) \\
&- \frac{3}{4\alpha^2}Y^h(\alpha)Z^v(\alpha) + \frac{1}{2\alpha}Y^h(Z^v(\alpha))\}X^v \\
&+ \frac{3}{4\alpha^3}g(X,Z)Y^h(\alpha)\bar{\nabla}\alpha - \frac{1}{2\alpha^2}g(X,Z)\bar{\nabla}_{Y^h}\bar{\nabla}\alpha,
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\bar{R}(X^v, Y^v)Z^h = &\frac{1}{\alpha^2}(R(X,Y)Z)^h - \frac{1}{\alpha^3}X^v(\alpha)(R(u,Y)Z)^h \\
&+ \frac{1}{\alpha^3}Y^v(\alpha)(R(u,X)Z)^h + \frac{1}{4\alpha^4}(R(u,X)R(u,Y)Z)^h \\
&- \frac{1}{4\alpha^4}(R(u,Y)R(u,X)Z)^h + \{-\frac{1}{4\alpha^3}(R(u,Y)Z)^h(\alpha) \\
&+ \frac{1}{2\alpha}Y^v(Z^h(\alpha)) - \frac{3}{4\alpha^2}Y^v(\alpha)Z^h(\alpha)\}X^v \\
&+ \{\frac{1}{4\alpha^3}(R(u,X)Z)^h(\alpha) - \frac{1}{2\alpha}X^v(Z^h(\alpha)) \\
&+ \frac{3}{4\alpha^2}X^v(\alpha)Z^h(\alpha)\}Y^v,
\end{aligned} \tag{18}
$$

and

$$
\begin{aligned}
\bar{R}(X^v, Y^v)Z^v = &\{\frac{1}{2\alpha}Y^v(Z^v(\alpha)) - \frac{1}{4\alpha^2}Y^v(\alpha)Z^v(\alpha)\}X^v \\
&+ \{-\frac{1}{2\alpha}X^v(Z^v(\alpha)) + \frac{1}{4\alpha^2}X^v(\alpha)Z^v(\alpha)\}Y^v \\
&+ \{\frac{3}{4\alpha^3}Y^v(\alpha)g(X,Z) - \frac{3}{4\alpha^3}X^v(\alpha)g(Y,Z)\}\bar{\nabla}\alpha \\
&+ \frac{1}{2\alpha^2}g(Y,Z)\bar{\nabla}_{X^v}\bar{\nabla}\alpha - \frac{1}{2\alpha^2}g(X,Z)\bar{\nabla}_{Y^v}\bar{\nabla}\alpha.
\end{aligned} \tag{19}
$$

Proof. We only prove the equation (16), the remaining ones are similar. Using curvature formula one gets

$$
\bar{R}(X^h, Y^v)Z^h = \bar{\nabla}_{X^h}\bar{\nabla}_{Y^v}Z^h - \bar{\nabla}_{Y^v}\bar{\nabla}_{X^h}Z^h - \bar{\nabla}_{[X^h,Y^v]}Z^h. \tag{20}
$$

From the equality (8) which is the calculated formula for the Levi-Civita connection of $\bar{g}$, one can get,

$$
\bar{\nabla}_{X^h}\bar{\nabla}_{Y^v}Z^h = \bar{\nabla}_{X^h}(\frac{1}{2\alpha^2}(R(u,Y)Z)^h + \frac{1}{2\alpha}Y^v(\alpha)Z^h - \frac{1}{2\alpha}Z^h(\alpha)Y^v). \tag{21}
$$

By taking acount (6), (7) and (10) in to (21), one can get,

$$\begin{aligned}
\bar{\nabla}_{X^h}\bar{\nabla}_{Y^v}Z^h &= \frac{u^i}{2\alpha^2}(\nabla_X R(\partial_i,Y)Z)^h - \frac{u^i}{2\alpha^2}(R(\nabla_X\partial_i,Y)Z)^h \\
&- \frac{3}{4\alpha^3}X^h(\alpha)(R(u,Y)Z)^h - \frac{1}{4\alpha^3}Z^h(\alpha)(R(u,Y)X)^h \\
&- \frac{1}{4\alpha^2}(R(X,R(u,Y)Z)u)^v - \frac{1}{4\alpha}Y^v(\alpha)(R(X,Z)u)^v \\
&+ \frac{1}{4\alpha^3}(R(u,Y)Z)^h(\alpha)X^h + \{-\frac{1}{4\alpha^2}X^h(\alpha)Y^v(\alpha) \\
&+ \frac{1}{2\alpha}X^h(Y^v(\alpha))\}Z^h + \{\frac{3}{4\alpha^2}X^h(\alpha)Z^h(\alpha) - \frac{1}{2\alpha}X^h(Z^h(\alpha))\}Y^v \\
&+ \frac{1}{2\alpha}Y^v(\alpha)(\nabla_X Z)^h - \frac{1}{2\alpha}Z^h(\alpha)(\nabla_X Y)^v \\
&+ \{-\frac{1}{4\alpha}Y^v(\alpha)g(X,Z) - \frac{1}{4\alpha^2}R(u,Y,Z,X)\}\bar{\nabla}\alpha.
\end{aligned} \tag{22}$$

Using the equation (6) in $\bar{\nabla}_{X^h}Z^h$ and putting the result in $\bar{\nabla}_{Y^v}\bar{\nabla}_{X^h}Z^h$ gives,

$$\begin{aligned}
\bar{\nabla}_{Y^v}\bar{\nabla}_{X^h}Z^h &= \bar{\nabla}_{Y^v}(\nabla_X Z)^h - \frac{1}{2\alpha^2}Y^v(\alpha)X^h(\alpha)Z^h + \frac{1}{2\alpha}Y^v(X^h(\alpha))Z^h \\
&+ \frac{1}{2\alpha}X^h(\alpha)\bar{\nabla}_{Y^v}Z^h - \frac{1}{2\alpha^2}Y^v(\alpha)Z^h(\alpha)X^h + \frac{1}{2\alpha}Y^v(Z^h(\alpha))X^h \\
&+ \frac{1}{2\alpha}Z^h(\alpha)\bar{\nabla}_{Y^v}X^h - \frac{1}{2}(R(X,Z)Y)^v - \frac{u^i}{2}\bar{\nabla}_{Y^v}(R(X,Z)\partial_i)^v \\
&- \frac{1}{2}g(X,Z)\bar{\nabla}_{Y^v}\bar{\nabla}\alpha.
\end{aligned} \tag{23}$$

By taking acount (8) in to (23), one can get,

$$\begin{aligned}\bar\nabla_{Y^v}\bar\nabla_{X^h}Z^h &= \frac{1}{2\alpha^2}(R(u,Y)\nabla_X Z)^h + \frac{1}{2\alpha}Y^v(\alpha)(\nabla_X Z)^h\\ &- \frac{1}{2\alpha}(\nabla_X Z)^h(\alpha)Y^v - \frac{1}{2\alpha^2}Y^v(\alpha)X^h(\alpha)Z^h\\ &+ \frac{1}{2\alpha}Y^v(X^h(\alpha))Z^h + \frac{1}{4\alpha^3}X^h(\alpha)(R(u,Y)Z)^h\\ &+ \frac{1}{4\alpha^2}X^h(\alpha)Y^v(\alpha)Z^h - \frac{1}{4\alpha^2}X^h(\alpha)Z^h(\alpha)Y^v\\ &- \frac{1}{2\alpha^2}Y^v(\alpha)Z^h(\alpha)X^h + \frac{1}{2\alpha}Y^v(Z^h(\alpha))X^h\\ &+ \frac{1}{4\alpha^3}Z^h(\alpha)(R(u,Y)X)^h + \frac{1}{4\alpha^2}Z^h(\alpha)Y^v(\alpha)X^h\\ &- \frac{1}{4\alpha^2}Z^h(\alpha)X^h(\alpha)Y^v - \frac{1}{2}(R(X,Z)Y)^v\\ &+ \frac{u^i}{4\alpha}Y^v(\alpha)(R(X,Z)\partial_i)^v + \frac{u^i}{4\alpha}(R(X,Z)\partial_i)^v(\alpha)Y^v\\ &- \frac{u^i}{4\alpha^2}R(X,Z,\partial_i,Y)\bar\nabla\alpha - \frac{1}{2}g(X,Z)\bar\nabla_{Y^v}\bar\nabla\alpha.\end{aligned} \tag{24}$$

According to the equality (8), $\bar\nabla_{(\nabla_X Y)^v}Z^h$ can be written as follow,

$$\begin{aligned}\bar\nabla_{(\nabla_X Y)^v}Z^h &= \frac{1}{2\alpha^2}(R(u,\nabla_X Y)Z)^h + \frac{1}{2\alpha}(\nabla_X Y)^v(\alpha)Z^h\\ &- \frac{1}{2\alpha}Z^h(\alpha)(\nabla_X Y)^v.\end{aligned} \tag{25}$$

By putting (22), (24) and (25) in (20) and after some calculations, the result can be achieved. ■

Theorem 5 *Let $\{E_1,\dots,E_n\}$ be an orthonormal locally frame on M. Then the Ricci operator $\bar Q$ of $\bar\nabla$ is determined by*

$$\begin{aligned}\bar Q(X^h) &= \frac{1}{\alpha}Q^h(X) + \{\frac{1}{4\alpha^2}||v\bar\nabla\alpha||^2 - \frac{1}{2\alpha}\Delta_{\bar g}\alpha\}X^h\\ &+ \frac{1}{2\alpha}\{h\bar\nabla_{X^h}h\bar\nabla\alpha - v\bar\nabla_{X^h}v\bar\nabla\alpha\}\\ &+ \frac{1}{2\alpha}\bar\nabla_{X^h}\bar\nabla\alpha - \frac{2n+1}{4\alpha^2}X^h(\alpha)\bar\nabla\alpha - \frac{1}{4\alpha^2}X^h(\alpha)h\bar\nabla\alpha\\ &+ \frac{1}{\alpha^2}X^h(\alpha)v\bar\nabla\alpha\\ &+ \Sigma_{i=1}^n\{\frac{3}{4\alpha^3}(R(u,R(X,E_i)u)E_i)^h + \frac{1}{4\alpha^2}(R(X,E_i)u)^v(\alpha)E_i^h\\ &- \frac{1}{4\alpha^3}(R(u,E_i)R(u,E_i)X)^h + \frac{1}{2\alpha}((\nabla_{E_i}R)(X,E_i)u)^v\\ &- \frac{3}{2\alpha^2}E_i^h(\alpha)(R(X,E_i)u)^v + \frac{1}{4\alpha^2}(R(u,E_i)X)^h(\alpha)E_i^v\},\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\bar{Q}(X^v) &= \{-\frac{1}{4\alpha^2}||v\bar{\nabla}\alpha||^2 - \frac{3}{4\alpha^2}||\bar{\nabla}\alpha||^2 + \frac{1}{2\alpha}\Delta_{\bar{g}}\alpha\}X^v \\
&+ \frac{1}{2\alpha}\{h\bar{\nabla}_{X^v}h\bar{\nabla}\alpha - v\bar{\nabla}_{X^v}v\bar{\nabla}\alpha\} \\
&- \frac{1}{2\alpha}\bar{\nabla}_{X^v}\bar{\nabla}\alpha + \frac{3-2n}{4\alpha^2}X^v(\alpha)\bar{\nabla}\alpha + \frac{3}{4\alpha^2}X^v(\alpha)v\bar{\nabla}\alpha \qquad (27) \\
&+ \Sigma_{i=1}^n\{-\frac{1}{2\alpha^3}((\nabla_{E_i}R)(u,X)E_i)^h + \frac{3}{2\alpha^4}E_i^h(\alpha)(R(u,X)E_i)^h \\
&- \frac{1}{4\alpha^4}(R(u,X)E_i)^h(\alpha)E_i^h + \frac{1}{4\alpha^3}(R(E_i,R(u,X)E_i)u)^v\}.
\end{aligned}$$

Proof. We shall give the proof of the equation (26). If X is a vector field on M then from the definition $\bar{Q}(X^h)$ we have

$$\bar{Q}(X^h) = \Sigma_{i=1}^n\{\frac{1}{\alpha}\bar{R}(X^h,E_i^h)E_i^h + \alpha\bar{R}(X^h,E_i^v)E_i^v\}. \qquad (28)$$

Using (14) and (17) gives us

$$\begin{aligned}
\bar{Q}(X^h) &= \frac{1}{\alpha}\Sigma_{i=1}^n\{(R(X,E_i)E_i)^h + \frac{1}{4\alpha^2}(R(u,R(X,E_i)u)E_i)^h \\
&+ \frac{1}{2\alpha^2}(R(u,R(X,E_i)u)E_i)^h + \{\frac{3}{4\alpha^2}E_i^h(\alpha)E_i^h(\alpha) - \frac{1}{2\alpha}E_i^h(E_i^h(\alpha))\}X^h \\
&+ \{-\frac{3}{4\alpha^2}X^h(\alpha)E_i^h(\alpha) + \frac{1}{2\alpha}X^h(E_i^h(\alpha)) + \frac{1}{4\alpha}(R(X,E_i)u)^v(\alpha)\}E_i^h \\
&+ \frac{1}{2}((\nabla_{E_i}R)(X,E_i)u)^v - \frac{1}{2\alpha}E_i^h(\alpha)(R(X,E_i)u)^v \\
&- \frac{1}{\alpha}E_i^h(\alpha)(R(X,E_i)u)^v + \{\frac{1}{4\alpha}X^h(\alpha)g(E_i,E_i) \\
&- \frac{1}{4\alpha}E_i^h(\alpha)g(X,E_i)\}\bar{\nabla}\alpha + \frac{1}{2}g(X,E_i)\bar{\nabla}_{E_i^h}\bar{\nabla}\alpha - \frac{1}{2}g(E_i,E_i)\bar{\nabla}_{X^h}\bar{\nabla}\alpha\} \\
&- \alpha\Sigma_{i=1}^n\{\frac{1}{4\alpha^4}(R(u,e_i)R(u,E_i)X)^h + \{\frac{1}{4\alpha^2}E_i^v(\alpha)E_i^v(\alpha) \\
&+ \frac{1}{2\alpha}e_i^v(E_i^v(\alpha))\}X^h + \{-\frac{1}{4\alpha^3}(R(u,E_i)X)^h(\alpha) \\
&- \frac{3}{4\alpha^2}X^h(\alpha)E_i^v(\alpha) + \frac{1}{2\alpha}X^h(E_i^v(\alpha))\}E_i^v \\
&+ \frac{3}{4\alpha^3}g(E_i,E_i)X^h(\alpha)\bar{\nabla}\alpha - \frac{1}{2\alpha^2}g(E_i,E_i)\bar{\nabla}_{X^h}\bar{\nabla}\alpha\}.
\end{aligned}$$

(29)

By setting the expressions

$$\Sigma_{i=1}^{n}(R(X,E_i)E_i)^h = Q^h(X),$$
$$\Sigma_{i=1}^{n}\frac{1}{\alpha}E_i^h(\alpha)E_i^h = h\bar{\nabla}\alpha,$$
$$\Sigma_{i=1}^{n}X^h(E_i^h(\alpha))E_i^h = X^h(\alpha)h\bar{\nabla}\alpha - \frac{1}{2}||h\bar{\nabla}\alpha||^2X^h + \alpha h\bar{\nabla}_{X^h}h\bar{\nabla}\alpha,$$
$$\Sigma_{i=1}^{n}g(E_i,E_i) = n,$$
$$\Sigma_{i=1}^{n}E_i^h(\alpha)g(X,E_i) = X^h(\alpha),$$
$$\Sigma_{i=1}^{n}g(X,E_i)E_i^h = X^h,$$
$$\Sigma_{i=1}^{n}\alpha E_i^v(\alpha)E_i^v = v\bar{\nabla}\alpha,$$

and

$$\Sigma_{i=1}^{n}X^h(E_i^v(\alpha)) = -\frac{1}{2\alpha^2}X^h(\alpha)v\bar{\nabla}\alpha + \frac{1}{\alpha}v\bar{\nabla}_{X^h}\bar{\nabla}\alpha,$$

in (29), we get the result; where $\bar{\nabla}\alpha = h\bar{\nabla}\alpha + v\bar{\nabla}\alpha$ and $\bar{\nabla}_{X^h}\bar{\nabla}\alpha = h\bar{\nabla}_{X^h}\bar{\nabla}\alpha + v\bar{\nabla}_{X^h}\bar{\nabla}\alpha$ are the decompositions of $\bar{\nabla}\alpha$ and $\bar{\nabla}_{X^h}\bar{\nabla}\alpha$ to the horizontal and vertical components, respectively. Moreover, $||h\bar{\nabla}\alpha||$ is the norm of the horizontal part of $\bar{\nabla}\alpha$ with respect to the metric $\bar{g}$. ■

The following proposition calculates the sectional curvatures of $\bar{g}$.

Proposition 6 *Suppose $\bar{K}$ represents the sectional curvature of $\bar{g}$. Let X and Y be locally orthonormal vector fields on M. Then $\bar{K}$ is given by*

$$\begin{aligned}\bar{K}(X^h,Y^h) &= \frac{1}{\alpha}K(X,Y) - \frac{3}{4\alpha^3}||R(X,Y)u||^2 + \frac{3}{4\alpha^3}Y^h(\alpha)Y^h(\alpha)\\ &\quad - \frac{1}{2\alpha^2}Y^h(Y^h(\alpha)) + \frac{1}{4\alpha^3}X^h(\alpha)X^h(\alpha)\\ &\quad - \frac{1}{2\alpha^2}\bar{g}(\bar{\nabla}_{X^h}\bar{\nabla}\alpha, X^h), \end{aligned} \tag{30}$$

$$\begin{aligned}\bar{K}(X^h,Y^v) &= \frac{1}{4\alpha^3}||R(u,Y)X||^2 - \frac{1}{4\alpha}Y^v(\alpha)Y^v(\alpha) - \frac{1}{2}Y^v(Y^v(\alpha))\\ &\quad - \frac{3}{4\alpha^3}X^h(\alpha)X^h(\alpha) + \frac{1}{2\alpha^2}\bar{g}(\bar{\nabla}_{X^h}\bar{\nabla}\alpha, X^h),\end{aligned}$$

and

$$\begin{aligned}\bar{K}(X^v,Y^v) &= \frac{1}{2}Y^v(Y^v(\alpha)) - \frac{1}{4\alpha}Y^v(\alpha)Y^v(\alpha) - \frac{3}{4\alpha}X^v(\alpha)X^v(\alpha)\\ &\quad + \frac{1}{2}\bar{g}(\bar{\nabla}_{X^v}\bar{\nabla}\alpha, X^v),\end{aligned}$$

where $K(X,Y)$ is the sectional curvature of g at the plane spanned by $\{X,Y\}$.

Proof. We only prove (30). Acoording to the definition of sectional curvature, we have

$$\bar{K}(\frac{X^h}{\sqrt{\alpha}}, \frac{Y^h}{\sqrt{\alpha}}) = \frac{1}{\alpha^2}\bar{g}(\bar{R}(X^h, Y^h)Y^h, X^h).$$

Using the equation (14) gives us

$$\begin{aligned}
\frac{1}{\alpha^2}\bar{g}(\bar{R}(X^h, Y^h)Y^h, X^h) = &\frac{1}{\alpha^2}\{\bar{g}((R(X,Y)Y)^h, X^h)\\
&+\frac{1}{4\alpha^2}\bar{g}((R(u, R(X,Y)u)Y)^h, X^h)\\
&+\frac{1}{2\alpha^2}\bar{g}((R(u, R(X,Y)u)Y)^h, X^h)\\
&+\{\frac{3}{4\alpha^2}Y^h(\alpha)Y^h(\alpha) - \frac{1}{2\alpha}Y^h(Y^h(\alpha))\}\bar{g}(X^h, X^h)\\
&+\frac{1}{4\alpha}X^h(\alpha)X^h(\alpha) - \frac{1}{2}\bar{g}(\bar{\nabla}_{X^h}\bar{\nabla}\alpha, X^h)\}.
\end{aligned}$$

Seting $\bar{g}((R(X,Y)Y)^h, X^h) = \alpha K(X,Y)$ and using the symmetries of R give us

$$\begin{aligned}
\bar{K}(\frac{X^h}{\sqrt{\alpha}}, \frac{Y^h}{\sqrt{\alpha}}) = &\frac{1}{\alpha}K(X,Y) - \frac{3}{4\alpha^3}||R(X,Y)u||^2\\
&+\frac{3}{4\alpha^3}Y^h(\alpha)Y^h(\alpha) - \frac{1}{2\alpha^2}Y^h(Y^h(\alpha))\\
&+\frac{1}{4\alpha^3}X^h(\alpha)X^h(\alpha) - \frac{1}{2\alpha^2}\bar{g}(\bar{\nabla}_{X^h}\bar{\nabla}\alpha, X^h).
\end{aligned}$$

■

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